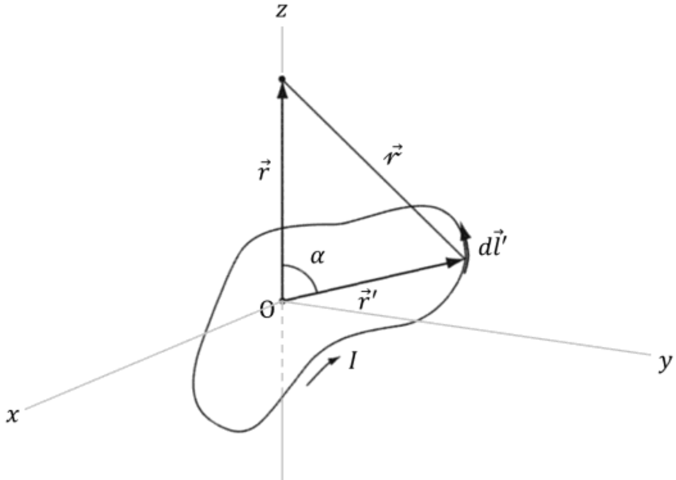
—Chapter 11—

Magnetic Fields in Matter

11-1 The Moment of Current Loop

A. THE MOMENT OF CURRENT LOOP

(1) The vector potential of a current loop at $\vec{r}$ is given by



$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathscr{r}} d\vec{l}'$$

where $\mathscr{r} = \left|\vec{r} - \vec{r}'\right|$, the distance from the current-element to the point $\mathscr{r}$. Write $1/\mathscr{r}$ in the form of a power series with Legendre polynomials,

$$\begin{aligned}\frac{1}{\mathscr{r}} &= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\alpha}} \\ &= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha) \\ &= \frac{1}{r} + \frac{r'}{r^2}\cos\alpha + \frac{1}{r^3}r'^2\frac{\left(3\cos^2\alpha - 1\right)}{2} + \cdots\end{aligned}$$

where α is the angle between $\vec{r}$ and $\vec{r}'$. Thus, we obtain

$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint \left[\frac{1}{r} + \frac{r'}{r^2} \cos\alpha + \frac{1}{r^3} r'^2 \frac{(3\cos^2\alpha - 1)}{2} + \cdots \right] d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi} \left[\underbrace{\frac{1}{r} \oint d\vec{l}'}_{①} + \underbrace{\frac{1}{r^2} \oint r' \cos\alpha \, d\vec{l}'}_{②} + \underbrace{\frac{1}{r^3} \oint r'^2 \frac{(3\cos^2\alpha - 1)}{2} d\vec{l}'}_{③} + \cdots \right]$$

The integral depends only on the closed loop. This power series is called the multipole expansion of the vector potential.

(2) Since the integral of the vector displacement around a closed loop is zero, i.e.,

$$\oint d\vec{l}' = 0$$

then we find that the first term, which is called the monopole, is always zero, i.e.,

$$\vec{A}_{①} = \frac{\mu_0 I}{4\pi} \frac{1}{r} \oint d\vec{l}' = 0$$

For the second term, which is called the dipole, we have

$$\oint r' \cos\alpha \, d\vec{l}' = \oint (\hat{r} \cdot \vec{r}') \, d\vec{l}'$$

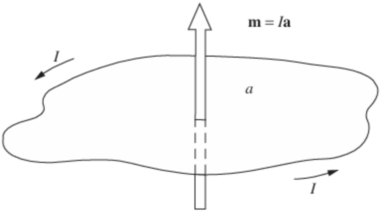
Then we can use a bit of vector calculus to simplify the dipole term. Let $\vec{c}$ be some constant vector. We obtain

$$\begin{aligned}
\vec{c} \cdot \oint (\hat{r} \cdot \vec{r}') \, d\vec{l}' &= \oint (\hat{r} \cdot \vec{r}') \vec{c} \cdot d\vec{l}' \\
&= \int \left(\nabla_{r'} \times (\hat{r} \cdot \vec{r}') \vec{c} \right) \cdot d\vec{a}' \\
&= \int \left(\nabla_{r'} (\hat{r} \cdot \vec{r}') \times \vec{c} \right) \cdot d\vec{a}' \\
&= \int \left(\hat{r} \times \vec{c} \right) \cdot d\vec{a}' \\
&= \int \left(d\vec{a}' \times \hat{r} \right) \cdot \vec{c}
\end{aligned}$$

$$\Rightarrow \oint (\hat{r} \cdot \vec{r}') \, d\vec{l}' = \int d\vec{a}' \times \hat{r}$$

We then define

$$\vec{m} = I\int d\vec{a}' = I\vec{a} \cdots\cdots \text{ called magnetic dipole moment}$$



The dipole term of the vector potential is

$$\vec{A}_{②} = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int d\vec{a}' \times \hat{r} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

For the third term, which is called the quadrupole, we have

$$\vec{A}_{③} = \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint\left(3\left(\hat{r}\cdot\vec{r}'\right)^2 - \left|\vec{r}'\right|^2\right)d\vec{l}'$$

(3) Thus, the vector potential becomes

$$\vec{A} = \frac{\mu_0}{4\pi}\left[\underbrace{\frac{\vec{m}\times\hat{r}}{r^2}}_{\text{dipole}} + \underbrace{\frac{I}{r^3}\oint\left(\frac{3\left(\hat{r}\cdot\vec{r}'\right)^2 - \left|\vec{r}'\right|^2}{2}\right)d\vec{l}'}_{\text{quadrupole}} + \cdots\right]$$

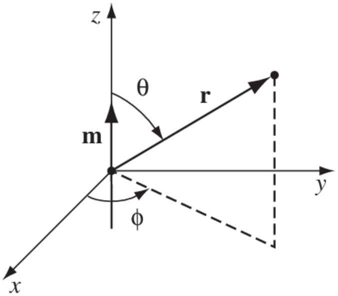
where the quantities $\vec{m}$ depends only on the loop.

B. THE MAGNETIC FIELD OF THE MOMENT

(1) The vector potential of a magnetic dipole is given by

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \quad \text{where } \vec{m} = I\vec{a} = \text{magnetic dipole moment}$$

If $\vec{m}$ is at the origin and points in the z-direction, the vector potential at point (r,θ,ϕ) is



Thus, we have

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

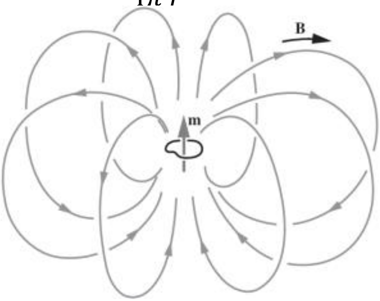
(2) The magnetic field of a magnetic dipole is

$$\vec{B} = \nabla \times \vec{A}$$

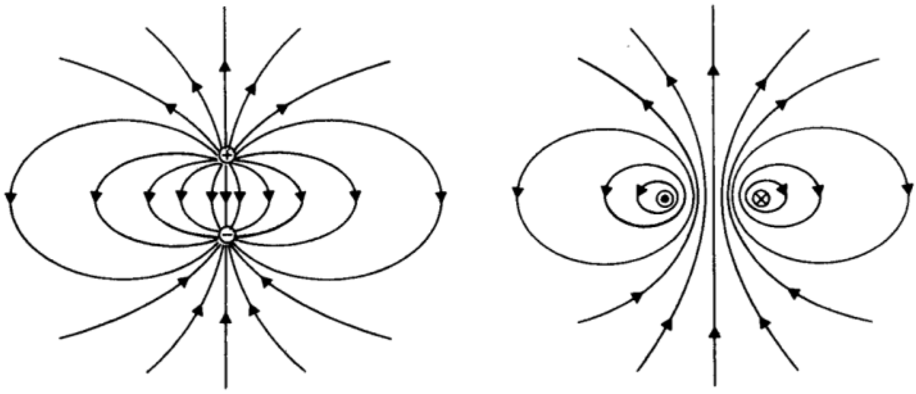
$$= \begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2} \end{vmatrix}$$

$$= \frac{\mu_0}{4\pi}m\begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix}$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



As we are far away a current ring, the field of a magnetic dipole would equal to the one of an electric dipole. Meanwhile, the magnetic field close to a current loop is entirely different from the electric field close to a pair of separated positive and negative charges.



Electric dipole. Magnetic dipole.

Since

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

the electric field starts at the positive charge and ends up at the negative charge. The electric field points down between the charges.

Since

$$\nabla \cdot \vec{B} = 0$$

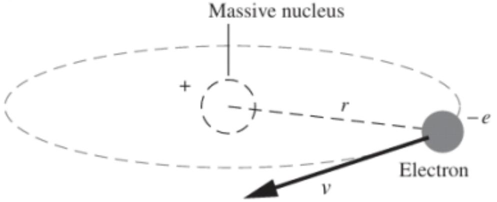
inside the current ring, the magnetic field does not end and points up.

11-2 Magnetization

A. MAGNETIC DIPOLE MOMENT OF AN ELECTRON

(1) Orbital magnetic dipole moment

We begin with one electron moving at constant speed on a circular path around the nucleus,



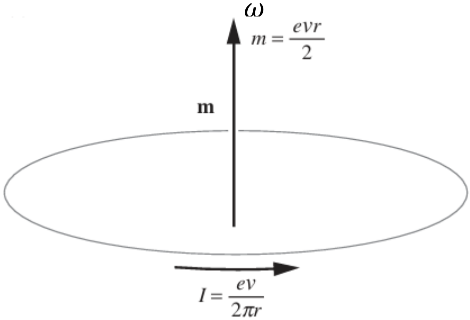
$$\vec{v} = \vec{\omega} \times \vec{r}$$

The orbital angular momentum is

$$\vec{L} = m_e \vec{r} \times \vec{v} = m_e r v \hat{z} = m_e r^2 \omega \hat{z}$$

Recall that the current is the charge passing a given point per unit time. The current is

$$I = -\frac{e}{T} = -\frac{e}{\frac{2\pi}{\omega}} = -\frac{e\omega}{2\pi} = -\frac{e\frac{v}{r}}{2\pi} = -\frac{ev}{2\pi r}$$



The magnetic dipole moment is therefore

$$\vec{m} = I\pi r^2 \hat{z} = -\frac{ev}{2\pi r}\pi r^2 \hat{z} = -\frac{evr}{2}\hat{z} = -\frac{e}{2m_e}\vec{L}$$

(2) Spin magnetic dipole moment

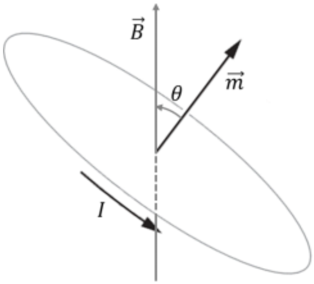
We define the spin angular momentum as

$$\vec{S}$$

The magnetic dipole moment of the electron spin is

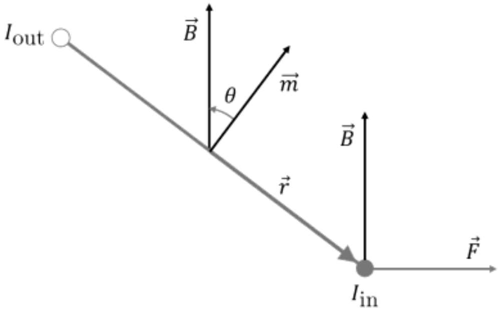
$$\vec{m} = -\frac{e}{2m_e}\vec{S}$$

(3) A magnetic dipole in an external field experiences a torque.



Consider the force on a small piece of the loop

$$d\vec{F} = dq\vec{v} \times \vec{B} = \lambda dl v\hat{v} \times \vec{B} = \lambda v dl\hat{v} \times \vec{B} = Id\vec{l} \times \vec{B}$$



The torque is

$$\vec{\tau} = \int \vec{r} \times d\vec{F} = \int rdF\sin\theta = \int I \underbrace{rdl}_{=da} B\sin\theta$$

Since

$$dl = rd\theta'$$

we obtain

$$\vec{\tau} = Ir^2 B\sin\theta \int_0^{\pi} d\theta' = \underbrace{I\pi r^2}_{=m} B\sin\theta = \vec{m} \times \vec{B}$$

This torque rotates the dipole unless it is placed parallel or anti-parallel to the field.

If we apply an external and opposite torque, it neutralizes the effect of this torque given by $\vec{\tau}$ and it rotates the dipole from the angle θ_0 to an angle θ at an infinitesimal angular speed without any angular acceleration. The amount of work done by the external torque can be given by

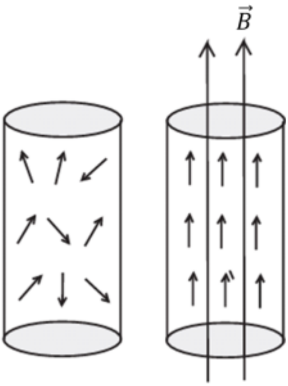
$$W = \int_{\theta_0}^{\theta} \vec{\tau} \cdot d\vec{s} = \int_{\theta_0}^{\theta} mB \sin\theta \, d\theta = -mB(\cos\theta_0 - \cos\theta)$$

Considering the initial angle to be the angle at which the potential energy is zero, the potential energy of the system can be given as,

$$U = -W = mB\left(\cos\frac{\pi}{2} - \cos\theta\right) = -mB\cos\theta = -\vec{m} \cdot \vec{B}$$

B. MAGNETIZATION

(1) With an applied field, randomly oriented permanent magnetic dipoles in a sample are aligned with an applied magnetic field.



The oriented dipoles will produce an additional magnetic field in the direction of $\vec{B}$. This phenomena is known as **paramagnetism**.

(2) Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out. But in the presence of a magnetic field, the orbital dipole moments points opposite or antiparallel to the magnetic field. This phenomena is known as **diamagnetism** which is really a quantum phenomenon.

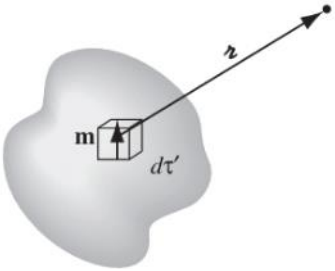
(3) A material is magnetized when a lot of little dipoles, point parallel or antiparallel along the direction of the field. A convenient measure of this effect is

$$\vec{M} = \vec{m}N \equiv \text{magnetic dipole moment per unit volume}$$

which is called the **magnetization**.

C. THE FIELD PRODUCED BY MAGNETIZED MATTER

(1) Suppose we have a piece of magnetized material - that is, an object containing a lot of microscopic dipoles $\vec{m}$ lined up.



The polarization $\vec{M}$ is given. The electric potential, at some external point, is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{M} \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

Since

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

we have

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{M} \times \nabla'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

Since

$$\nabla' \times \left(\frac{1}{\mathscr{r}}\vec{M}\right) = \frac{1}{\mathscr{r}}\nabla' \times \vec{M} - \vec{M} \times \nabla'\frac{1}{\mathscr{r}}$$

we have

$$\vec{A} = \frac{\mu_0}{4\pi}\left[\int \frac{1}{\mathscr{r}}\left(\nabla' \times \vec{M}\right) d\tau' - \int \nabla' \times \left(\frac{\vec{M}}{\mathscr{r}}\right) d\tau'\right]$$

Let $\vec{c}$ be some constant vector. We can find Gauss's divergence theorem in the form as

$$\int_{\mathcal{V}} \nabla \cdot \left(\vec{M} \times \vec{c}\right) d\tau = \oint_{\mathcal{S}} \left(\vec{M} \times \vec{c}\right) \cdot d\vec{a}$$

Using vector product rules of ∇ and triple products identity:

$$\nabla \cdot \left(\vec{M} \times \vec{c}\right) = \vec{c} \cdot \left(\nabla \times \vec{M}\right) - \vec{M} \cdot (\nabla \times \vec{c}) = \vec{c} \cdot \left(\nabla \times \vec{M}\right)$$

$$\left(\vec{M} \times \vec{c}\right) \cdot d\vec{a} = -\vec{c} \cdot \left(\vec{M} \times d\vec{a}\right)$$

Thus, we have

$$\int_{\mathcal{V}} \vec{c}\cdot\left(\nabla\times\vec{M}\right)d\tau = -\oint_{\mathcal{S}} \vec{c}\cdot\left(\vec{M}\times d\vec{a}\right) \Rightarrow \int_{\mathcal{V}}\left(\nabla\times\vec{M}\right)d\tau$$
$$= -\oint_{\mathcal{S}} \vec{M}\times d\vec{a}$$

Then we express the vector potential as

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}\left(\nabla'\times\vec{M}\right)d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\vec{M}}{\mathscr{r}}\times d\vec{a}'$$

$$= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}\underbrace{\left(\nabla'\times\vec{M}\right)}_{\substack{\text{volume}\\ \text{current}\\ \text{density}}}d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\overbrace{\left(\vec{M}\times\hat{n}'\right)}^{\substack{\text{surface}\\ \text{current}\\ \text{density}}}}{\mathscr{r}}da'$$

We then let

$\vec{J}_b = \nabla'\times\vec{M}$ ⋯⋯ volume current density

$\vec{K}_b = \vec{M}\times\hat{n}'$ ⋯⋯ surface current density

$\vec{J}_b$ and $\vec{K}_b$ are called bound current.

(2) The magnetic field produced by a magnetized matter is

$$\vec{B} = \nabla\times\vec{A}$$
$$= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \nabla\times\frac{\vec{J}_b}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \nabla\times\frac{\vec{K}_b}{\mathscr{r}}da'$$
$$= -\frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\hat{\mathscr{r}}\times\vec{J}_b}{\mathscr{r}^2}d\tau' - \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\hat{\mathscr{r}}\times\vec{K}_b}{\mathscr{r}^2}da'$$
$$= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\vec{J}_b\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\vec{K}_b\times\hat{\mathscr{r}}}{\mathscr{r}^2}da'$$

Consider a case of a uniformly magnetized matter, i.e., $\vec{M}$ = constant.
Since

$$\vec{J}_b = \nabla'\times\vec{M} = 0$$

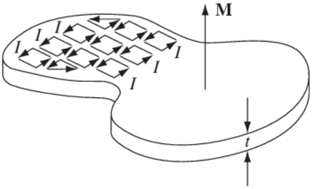
we obtain

$$\vec{B} = \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\vec{K}_b da'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 K_b}{4\pi}\oint_{\mathcal{S}} \frac{d\vec{a}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

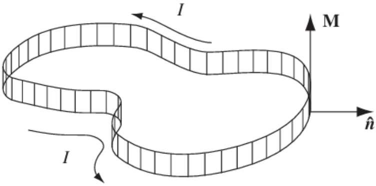
This is the Biot-Savart law for a surface current.

Physical interpretation:

Suppose that there is a uniformly magnetized slab. The slab consists of many dipoles represented by tiny current loops.

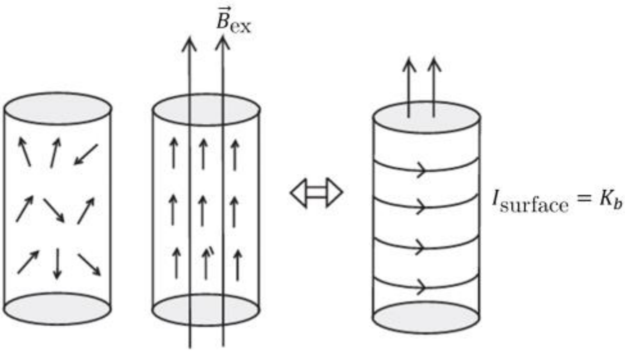


All the "internal" currents cancel, but at the edge there is no adjacent loop to do the canceling. The whole thing, then, is equivalent to a single ribbon of current flowing around the boundary.



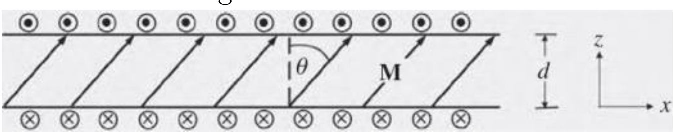
$\vec{K}_b = \vec{M} \times \hat{n}$

The magnetic field at any point outside the magnetized matter is the same as the field at the corresponding point in the neighborhood of a surface current flowing around the matter.



EXAMPLES:

1. Find the magnetic field produced by an infinite slab of matter with uniform magnetization.



ANSWER:

$\vec{J}_b = \nabla \times \vec{M} = 0$

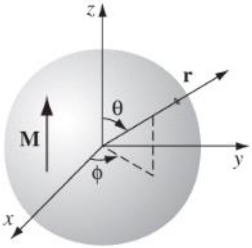
$\vec{K}_b = \vec{M} \times \hat{n}$

For the top surface current, we have $\vec{K}_b = -M \sin\theta\, \hat{y}$.

Using Ampère's law, we obtain the magnetic field inside the slab produced by the top surface current:

$$\oint_{\mathcal{C}} \vec{B} \cdot d\vec{s} = B_x l = \mu_0 K_b l \Rightarrow \vec{B} = \mu_0 M \sin\theta\, \hat{x}$$

2. Find the magnetic field of a uniformly magnetized sphere.



ANSWER:

Choosing the z axis along the direction of $\vec{M}$, we have

$\vec{J}_b = \nabla \times \vec{M} = 0$

$\vec{K}_b = \vec{M} \times \hat{n} = M \sin\theta\, \hat{\phi}$

The field is that of a surface current.

Since a rotating spherical shell, of uniform surface charge σ, corresponds to a surface current density

$\vec{K} = \sigma\vec{v} = \sigma R\omega \sin\theta\, \hat{\phi} \Rightarrow \sigma R\omega = M$

the field of a uniformly magnetized sphere is identical to the field of a spinning spherical shell.

Since the vector potential of a spinning spherical shell is [c.f.6-2]

$$\vec{A} = \begin{cases} \dfrac{\mu_0 \sigma R\omega}{3} r \sin\theta\, \hat{\phi} = \dfrac{\mu_0 M}{3} r \sin\theta\, \hat{\phi} & , \quad r \leq R \\ \dfrac{\mu_0 \sigma R^4 \omega}{3} \dfrac{\sin\theta}{r^2} \hat{\phi} = \dfrac{\mu_0 R^3 M}{3} \dfrac{\sin\theta}{r^2} \hat{\phi} = \dfrac{\mu_0 m}{4\pi} \dfrac{\sin\theta}{r^2} \hat{\phi}, & \quad r \geq R \end{cases}$$

Thus, the field of the spinning spherical shell:

- For $r \leq R$:

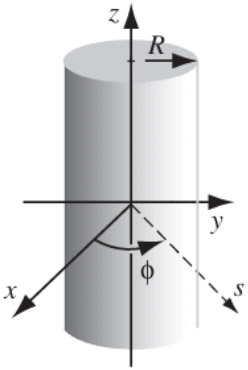
$$\vec{B} = \nabla \times \vec{A}$$
$$= \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta \frac{\mu_0 M}{3} r\sin\theta \end{vmatrix}$$
$$= \frac{\mu_0 M}{3} \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix}$$
$$= \frac{2\mu_0 M}{3}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$
$$= \frac{2}{3}\mu_0 M\hat{z}$$

The field inside the spherical shell is uniform.

- For $r \geq R$:

$$\vec{B} = \nabla \times \vec{A}$$
$$= \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta \frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^2} \end{vmatrix}$$
$$= \frac{\mu_0 m}{4\pi} \begin{vmatrix} \frac{1}{r^2 \sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix}$$
$$= \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

3. An infinitely long circular cylinder carries a uniform magnetization $\vec{M}$ parallel to its axis. Find the magnetic field (due to $\vec{M}$) inside and outside the cylinder.



ANSWER:

$\vec{J}_b = \nabla \times \vec{M} = 0$

$\vec{K}_b = \vec{M} \times \hat{n} = M\hat{\phi}$

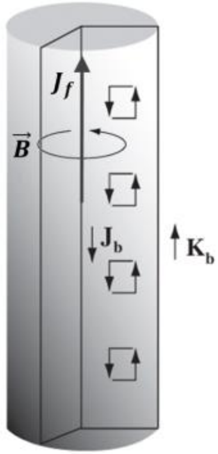
The field is that of a surface current. Using Ampère's law, we have

$$\vec{B} = \begin{cases} \mu_0 K_b = \mu_0 M \hat{z}, & r \leq R \\ 0 \quad , & r \geq R \end{cases}$$

11-3 Magnetic Susceptibility

A. AMPÈRE'S LAW WITH MAGNETIZED MATTER

(1) Consider a long copper rod carries a uniformly distributed (free) current $\vec{J}_f$. Since copper is weakly diamagnetic, so the dipoles will line up opposite to the field.



This results in a bound current $\vec{J}_b$ running antiparallel to $\vec{J}_f$, within the wire, and $\vec{K}_b$ parallel to $\vec{J}_f$ along the surface.
The total current can be written as

$$\vec{J} = \vec{J}_f + \vec{J}_b$$

(2) Ampère's law can be written

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \mu_0 \left(\vec{J}_f + \vec{J}_b\right) = \mu_0 \left(\vec{J}_f + \nabla \times \vec{M}\right)$$

Collecting together the two curls:

$$\nabla \times \left(\vec{B} - \mu_0 \vec{M}\right) = \mu_0 \vec{J}_f \Rightarrow \nabla \times \left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{J}_f$$

We then define a vector function $\vec{H}$ as

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$$

In terms of $\vec{H}$, then, Ampère's law reads

$$\nabla \times \vec{H} = \vec{J}_f$$

Using Stokes' theorem,

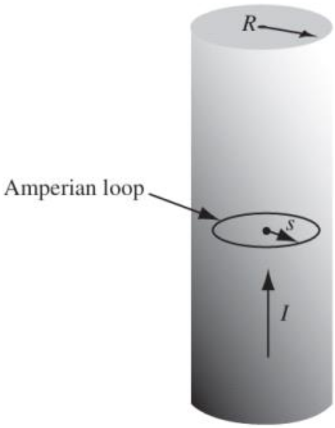
$$\int_{\mathcal{S}} \left(\nabla \times \vec{H}\right) \cdot d\vec{a} = \oint_{\mathcal{C}} \vec{H} \cdot d\vec{s} \text{ and } \int_{\mathcal{S}} \vec{J}_f \cdot d\vec{a} = I_f$$

we obtain Ampère's law in integral form:

$$\oint_{\mathcal{C}} \vec{H} \cdot d\vec{s} = I_f$$

EXAMPLES:

1. A long copper rod of radius R carries a uniformly distributed (free) current I. Find $\vec{H}$ inside and outside the rod.



ANSWER:

Inside the wire:

$$\oint_{\mathcal{C}} \vec{H} \cdot d\vec{s} = H2\pi s = I_f = I\frac{\pi s^2}{\pi R^2} \Rightarrow \vec{H} = \frac{Is}{2\pi R^2}\hat{\phi}$$

Outside the wire:

$$\oint_{\mathcal{C}} \vec{H} \cdot d\vec{s} = H2\pi s = I_f = I \Rightarrow \vec{H} = \frac{I}{2\pi s}\hat{\phi}$$

Since $\vec{M} = 0$ outside the wire, so we have

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

2. An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis,

   $\vec{M} = kr\hat{z}$

   There is no free current anywhere.

   (a) Locate all the bound currents, and calculate the field inside the cylinder they produce.

(b) Use Ampère's law to find $\vec{H}$ inside the cylinder, and then get $\vec{B}$.

ANSWER:

(a)

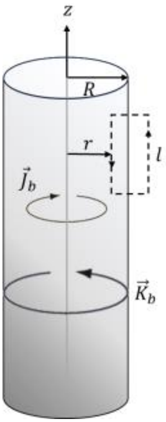
The volume current is

$$\vec{J}_b = \nabla \times \vec{M} = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\phi} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & kr \end{vmatrix} = -k\hat{\phi}$$

The surface current is

$$\vec{K}_b = \vec{M} \times \hat{n} = kR\hat{\phi}$$

The bound currents produce a solenoidal field. The field outside the cylinder will be equal to zero and the field inside the cylinder will be directed along the z axis. Its magnitude can be obtained using Ampère's law:



$$\oint_C \vec{B}\cdot d\vec{s} = -Bl$$
$$= \mu_0\left[-K_b l - \int_r^R \vec{J}_b \cdot l d\vec{r}\right]$$
$$= \mu_0[-kRl + kl(R-r)]$$
$$= -\mu_0 krl$$

Thus, the field inside the cylinder is

$$\vec{B} = \mu_0 kr\hat{z}$$

(b)

$$\oint_C \vec{H}\cdot d\vec{s} = Hl = 0 \ \because \ \text{no free current}$$
$$\Rightarrow \vec{H} = 0 = \frac{1}{\mu_0}\vec{B} - \vec{M}$$
$$\Rightarrow \vec{B} = \mu_0\vec{M} = \mu_0 kr\hat{z}$$

B. LINEAR MEDIA

(1) Since in the laboratory, the current is the thing read on the ammeter which determines $\vec{H}$. Thus, we assume that in the presence of a field $\vec{H}$, the magnetization is proportional to the field.

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is called the magnetic susceptibility.

χ_m is positive for paramagnets and negative for diamagnetis where $\chi_m \sim 10^{-5}$ for most ordinary materials.

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.7×10^{-4}	Oxygen (O_2)	1.7×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.2×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.0×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.7×10^{-4}
Carbon Dioxide	-1.1×10^{-8}	Liquid Oxygen (−200° C)	3.9×10^{-3}
Hydrogen (H_2)	-2.1×10^{-9}	Gadolinium	4.8×10^{-1}

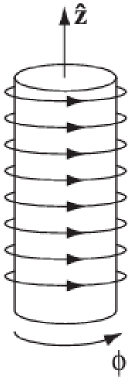
(2) Thus, we obtain

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \chi_m \vec{H} \Rightarrow \vec{H} = \frac{1}{\mu_0(1+\chi_m)}\vec{B} = \frac{1}{\mu}\vec{B}$$

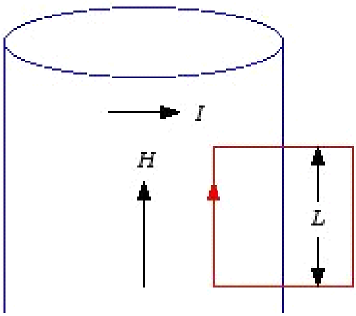
where μ is called the permeability of the material, μ_0 is called the permeability of free space.

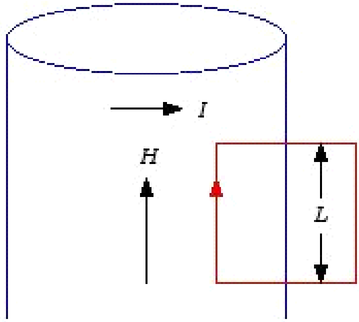
EXAMPLES:

1. An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.



ANSWER:





The line integral of $\vec{H}$ around the loop is equal to

$$\oint_C \vec{H}\cdot d\vec{s} = HL$$

The free current intercepted by the Ampèrian loop is equal to

$$I_{\text{enc}} = nIL$$

The $\vec{H}$ field is

$$\vec{H} = nI\hat{z}$$

The magnetic field inside the solenoid is equal to

$$\vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu_0(1+\chi_m)nI\hat{z}$$

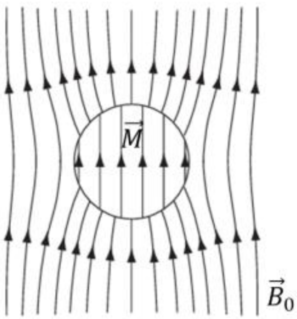
The magnetization of the material is equal to

$$\vec{M} = \chi_m\vec{H} = \chi_m nI\hat{z}$$

and is uniform. Then, the bound surface current is equal to

$$\vec{K}_b = \vec{M}\times\hat{n} = \chi_m\vec{H}\times\hat{n} = \chi_m nI\hat{\phi}$$

2. A sphere of linear magnetic material is placed in an otherwise uniform magnetic field $\vec{B}_0$. Find the total field inside the sphere.



ANSWER:

- Method I:

Inside the magnetized sphere, the total field is

$$\vec{B}' = \vec{B}_0 + \vec{B}_{\text{in}}$$

Since

$$\vec{M} = \chi_m \vec{H}' = \frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}'$$

and [c.f.11-2]

$$\vec{B}_{\text{in}} = \frac{2\mu_0}{3}\vec{M}$$

we obtain

$$\frac{\mu_0(1+\chi_m)}{\chi_m}\vec{M} = \vec{B}_0 + \frac{2\mu_0}{3}\vec{M} \Rightarrow \vec{M} = \frac{1}{\frac{(1+\chi_m)}{\chi_m} - \frac{2}{3}}\frac{\vec{B}_0}{\mu_0} = \frac{3\chi_m}{3+\chi_m}\frac{\vec{B}_0}{\mu_0}$$

The total field inside the sphere can be also expressed as

$$\vec{B}' = \vec{B}_0 + \frac{2\mu_0}{3}\frac{3\chi_m}{3+\chi_m}\frac{\vec{B}_0}{\mu_0} = \frac{3(1+\chi_m)}{3+\chi_m}\vec{B}_0$$

- Method II:

The external field $\vec{B}_0$ will magnetize the sphere:

$$\vec{M}_0 = \chi_m \vec{H}_0 = \frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}_0$$

This magnetization will produce a uniform magnetic field inside the sphere [c.f.11-2]

$$\vec{B}_1 = \frac{2}{3}\mu_0 \vec{M}_0 = \frac{2}{3}\mu_0 \frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}_0 = \frac{2\chi_m}{3(1+\chi_m)}\vec{B}_0$$

This additional magnetic field magnetizes the sphere by an additional amount:

$$\begin{aligned}\vec{M}_1 &= \frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}_1 \\ &= \frac{\chi_m}{\mu_0(1+\chi_m)}\frac{2\chi_m}{3(1+\chi_m)}\vec{B}_0 \\ &= \frac{2}{3\mu_0}\left(\frac{\chi_m}{(1+\chi_m)}\right)^2 \vec{B}_0\end{aligned}$$

This additional magnetization produces an additional magnetic field inside the sphere:

$$\vec{B}_2 = \frac{2}{3}\mu_0 \vec{M}_1 = \frac{2}{3}\mu_0 \frac{2\chi_m^2}{3\mu_0(1+\chi_m)^2}\vec{B}_0 = \left(\frac{2\chi_m}{3(1+\chi_m)}\right)^2 \vec{B}_0$$

The total magnetic field inside the sphere is therefore equal to

$$\vec{B} = \sum_{n=0} \left(\frac{2\chi_m}{3(1+\chi_m)} \right)^n \vec{B}_0 = \frac{1}{1 - \frac{2\chi_m}{3(1+\chi_m)}} \vec{B}_0 = \frac{3(1+\chi_m)}{3+\chi_m} \vec{B}_0$$

C. BOUNDARY-VALUE PROBLEMS WITH DIELECTRICS

(1) According to Helmholtz theorem, the magnetic field $\vec{B}$ is uniquely determined by

$$\nabla \times \vec{B} = \mu_0 \vec{J} \text{ and } \nabla \cdot \vec{B} = 0$$

Since the divergence of $\vec{H}$ is not always zero:

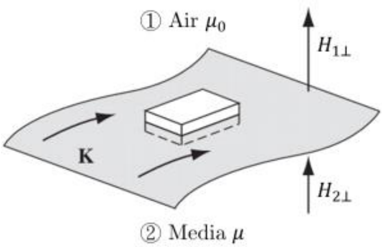
$$\nabla \cdot \vec{H} = \frac{1}{\mu_0} \nabla \cdot \vec{B} - \nabla \cdot \vec{M} = -\nabla \cdot \vec{M} \stackrel{?}{=} 0$$

$\vec{H}$ cannot be uniquely determined by the free charge only as

$$\nabla \times \vec{H} = \vec{J}_{\text{free}}$$

Thus, we need boundary conditions on $\vec{H}$ at various dielectric surfaces.

(2) We choose a Gaussian surface for a very tiny area $d\vec{a}$ and let the thickness go to zero.



Since

$$\oint_S \vec{H} \cdot d\vec{a} = \int_V \left(\nabla \cdot \vec{H} \right) d\tau = -\int_V (\nabla \cdot M)\, d\tau = -\oint_S \vec{M} \cdot d\vec{a}$$

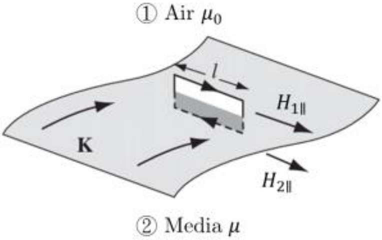
we obtain

$$\underbrace{H_{1\perp} da}_{\text{Air}} - \underbrace{H_{2\perp} da}_{\text{media}} = -\underbrace{M_{1\perp} da}_{\text{Air}} + \underbrace{M_{2\perp} da}_{\text{media}}$$

$\Rightarrow H_{1\perp} - H_{2\perp} = -M_{1\perp} + M_{2\perp}$

$H_\perp$ is also discontinuous across the surface of the magnetized matter.

We then can choose a closed loop such that the width goes to zero as



Thus, we obtain

$$\oint_C \vec{H} \cdot d\vec{s} = \underbrace{H_{1\parallel} ds}_{\text{Air}} - \underbrace{H_{2\parallel} ds}_{\text{media}} = K_{\text{free}} ds \Rightarrow \underbrace{H_{1\parallel}}_{\text{media ①}} - \underbrace{H_{2\parallel}}_{\text{media ②}} = \vec{K}_{\text{free}} \times \hat{n}$$

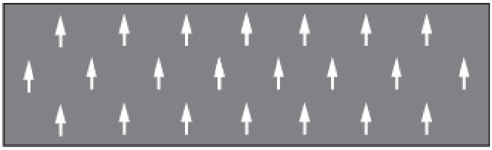
$H_\parallel$ is discontinuous across the surface of the magnetized matter. For linear media $\vec{H} = \vec{B}/\mu$, we have

$$\frac{B_{1\parallel}}{\underbrace{\mu_1}_{\text{media ①}}} - \frac{B_{2\parallel}}{\underbrace{\mu_2}_{\text{media ②}}} = \vec{K}_{\text{free}} \times \hat{n}$$

11-4 Ferromagnetism

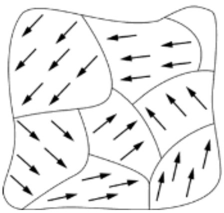
A. DOMAINS

(1) Ferromagnets—which are emphatically not linear—require no external fields to sustain the magnetization; the alignment is "frozen in."



In a ferromagnet, each dipole "likes" to point in the same direction as its neighbors. The reason for this preference is essentially quantum mechanical.

(2) The alignment occurs in relatively small patches, called domains.

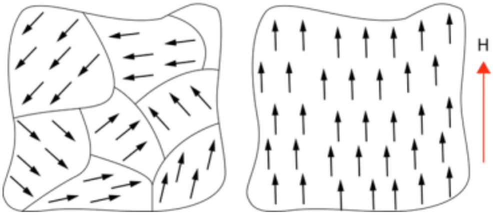


Each domain contains billions of dipoles, all lined up, but the domains themselves are randomly oriented.

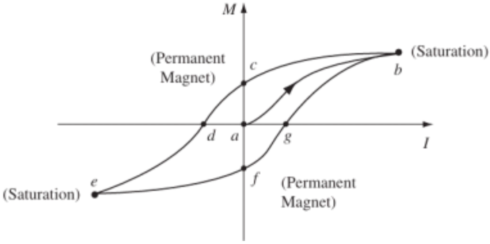
(3) Near the walls the spin of atoms of one region get slowly oriented in the favorable direction of neighboring domain.

B. HYSTERESIS [histəˈr ē səs] LOOP

(1) Domains parallel to the field grow, and the others shrink. If the field is strong enough, one domain takes over entirely, and the iron is said to be saturated.



(2) Starting with unmagnetized iron, $M = 0$, increasing H causes M to rise in a conspicuously nonlinear way, slowly at first, then more rapidly, then very slowly, finally flattening off.



If we now slowly decrease the current, thus lowering H, the curve does not retrace itself. Instead, we find the behavior is irreversible, which is called hysteresis. It is largely due to the domain boundary movements being partially irreversible.